Write-up 3: Quadratics Exploration in the x-b Plane

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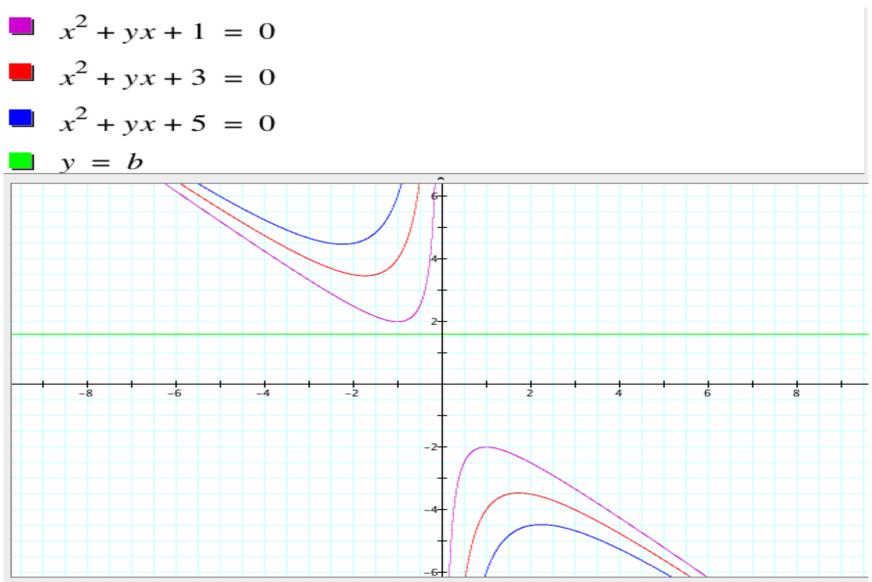
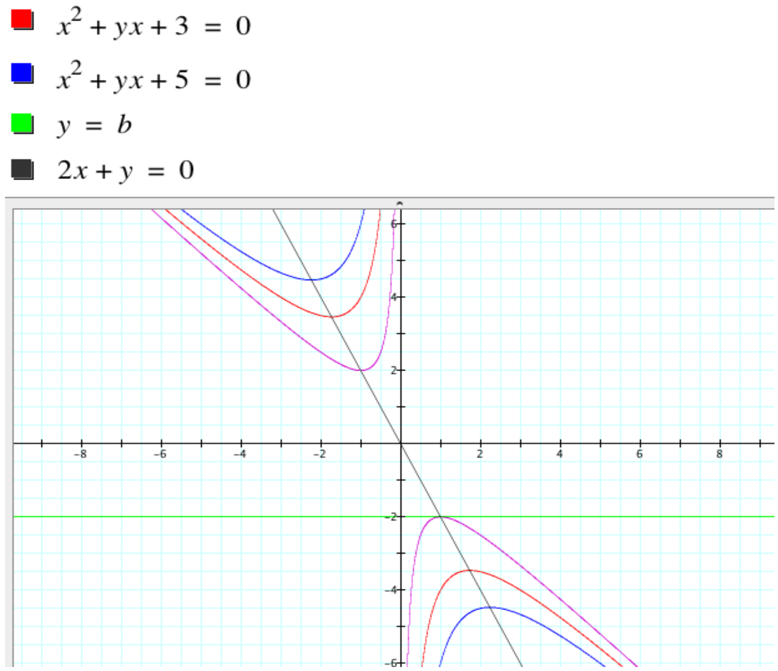
We will here analyze quadratics $x^2 + bx + c = 0$ in the $x - b$ plane by renaming b as y and graphing as such:



In this graph we have also considered various values of y by graphing the line $y = b$ where b varies. As the coefficient b changes, the line moves such that it intersects $x^2 + yx + 1 = 0$ zero, one, or two times. The x-coordinates of the one or two intersections (x, b) correspond to the roots of $x^2 + bx + 1 = 0$. For example, when $y = b = 2$, the graphs intersect at only $(1, -2)$; not coincidentally, $x = 1$ is the only root of $x^2 - 2x + 1 = 0$.
Now, let us consider the line $2x + y = 0$.



Observe how $2x + y = 0$ seems to split the hyperbola in half. For example, consider $x^2 + yx + 3 = 0$ when $y = b = -4$. The roots are $x = 1$, $x = 3$ and the intersections of the the quadratic and $y = b = -4$ are $(1, -4)$ and $(3, -4)$. At the same time, when $y = -4$, the line $2x + y = 0$ passes through $(2, -4)$ - the point exactly halfway between the two roots.

If we consider what we know about the roots of quadratics, though, this phenomenon is quite simple. Given any quadratic $ax^2 + bx + c = 0$, the quadratic formula (derived by completing the square using the arbitrary variables) tells us that the roots are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{and} x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

If we take $a = 1$ and average the two roots, we get

$$\frac{1}{2}(\frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2})$$
$$= \frac{1}{2}(\frac{-2b}{2})$$
$$= \frac{-b}{2}$$

Now, rewriting the line $2x + y = 0$ as $2x + b = 0$ and simplifying, we similarly get

$$x = \frac{-b}{2}$$

Therefore, the lines necessarily intersect at $(\frac{-b}{2}, b)$ halfway between the two roots of the quadratic.